1. (40 pts) Let $f(s,x) = \dfrac{2+3s\sqrt[3]{x}}{(1+s\sqrt[3]{x})(1+x)}, x \geq 0$.

(a) Let $g_k(x) = f(k,x), k = 1,2,3,\ldots$ Find $\lim_{k\to\infty} g_k(x)$ for $x \geq 0$.

(b) Does $\{g_k(x)\}$ converge uniformly on $(0,1]$?

(c) Find $\lim_{s\to 0^+} \dfrac{1}{s}\int_0^s f(s,x)\,dx$.

(d) Find $\lim_{s\to\infty} \dfrac{1}{\ln s}\int_0^s f(s,x)\,dx$.

(e) Show that there exists $\hat{s} > 0$ such that

$$\hat{s} = \int_0^{\hat{s}} f(\hat{s},x)\,dx.$$

(f) Let $s_0 = 0.01, s_{k+1} = \int_0^{s_k} f(s_k,x)\,dx$, $k = 0,1,2,3,\ldots$ Show that $\lim_{k\to\infty} s_k$ exists.

2. (30 pts) Assume that $D = [0,1]\times[0,1]$ and E is a closed subset of D. For $x = (x_1,x_2)$, let $|x| = \sqrt{x_1^2+x_2^2}$ denote the Euclidean norm of x in $\mathbb{R}^2$.

(a) Let $d(x) = \inf_{y\in E}|x-y|$. Show that for each $x \in D$, there exists $\hat{y} \in E$ such that $d(x) = |x-\hat{y}|$.

(b) Show that $d(x)$ is a continuous function on D.

(c) Show that $\sup_{x\in D} d(x) \leq \inf_{y\in E}[\sup_{x\in D}|x-y|]$.

(d) Find an example E such that $\sup_{x\in D}[\inf_{y\in E}|x-y|] < \inf_{y\in E}[\sup_{x\in D}|x-y|]$ holds.

3. (30 pts) Let $f(x)$ and $g(x,y)$ be C^2 functions.

(a) Show that $\lim_{h\to 0}\dfrac{f(3h)-3f(h)+2f(0)}{h^2} = 3f''(0)$.

(b) Assume $f(x+2h)-2f(x+h)+f(x) = 0$ for all x and h. Prove that $f(x) = ax+b$ for some constants a and b.

(c) Show that $\lim_{h\to 0}\dfrac{g(h,h)-g(h.0)-g(0,h)+g(0,0)}{h^2} = \dfrac{\partial^2 g}{\partial x\partial y}(0,0)$.

(d) Assume $g(x+h,y+h)-g(x+h.y)-g(x,y+h)+g(x,y) = 0$ for all x, y and h, $g(x,0) = x^2+1$ and $g(0,y) = \cos y$. Find the function $g(x,y)$.

試題隨卷繳回